Chaos in a two-spin system with applied magnetic field

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We observe both classical and quantum chaos in a system of two interacting magnetic moments in an applied magnetic field. Chaos in the classical system, as seen in the Poincaré surface of section, and chaos in the quantum system, as indicated by the changing patterns of the quantum web, appear at the same value of the applied magnetic field parameter. The signatures of chaos disappear in both the classical and quantum systems at a larger value of the field parameter. [S1063-651X(98)04502-4]

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The effects on a quantum system of a transition to chaos in its classical counterpart have proved important in explaining several physical phenomena [1–6]. In this paper we focus on these effects in a two-spin system. In a series of articles [7–9], Magyari *et al.* and Srivastava *et al.* have provided insight into the origin of the phenomenon of level repulsion as well as demonstrated the appearance of classical and quantum chaotic effects in a model of the interaction between two neighboring atoms similar to that in a ferromagnetic lattice. Here we investigate the transition to chaos for the case in which a magnetic field is applied to the two neighboring atoms. This effect may prove important in understanding the behavior of thin magnetic films.

Our model Hamiltonian for two localized quantum spins

$$\hat{H}=\hat{H}_{xy}+\hat{H}_{\lambda}=-J(\hat{\sigma}_{1x}\hat{\sigma}_{2x}+\hat{\sigma}_{1y}\hat{\sigma}_{2y})+\lambda(\hat{\sigma}_{1x}+\hat{\sigma}_{2x}) \tag{1}$$

consists of an xy model interaction between the two spins and an interaction with an applied magnetic field in the x direction, with the coupling between the spin magnetic moments and the applied magnetic field parametrized by λ. The Hamiltonian $\hat{H}$ of the system is a constant of the motion When $\lambda=0$, this system has a second constant of the motion,

$$\hat{\sigma}_z=\hat{\sigma}_{1z}+\hat{\sigma}_{2z}, \tag{2}$$

as may be verified using the angular momentum commutation relations $[\hat{\sigma}_{ix},\hat{\sigma}_{iy}]=i\hbar\hat{\sigma}_{iz}$ for spins $i=1,2$ and cyclic permutations of x,y,z. The classical limit of this system is established by allowing $\hbar\rightarrow 0$ and $\sigma\rightarrow\infty$ such that $\hbar\sqrt{\sigma(\sigma+1)}=S$, where S is some fixed finite value. This produces a classical system of two angular momenta $\vec{S}_1$ and $\vec{S}_2$ interacting via the Hamiltonian $H(\vec{S}_1,\vec{S}_2)=-J(S_{1x}S_{2x}+S_{1y}S_{2y})+\lambda(S_{1x}+S_{2x})$ and obeying the mean-field equation of motion $d\vec{S}_l/dt=-\vec{S}_l\times\partial H/\partial\vec{S}_l$. The classical spins $\vec{S_l}=S(\sin\theta_l\cos\phi_l,\sin\theta_l\sin\phi_l,\cos\theta_l)$ can also be expressed in terms of canonical coordinates $p_l=S\cos\theta_l$ and $q_l=\phi_l$, so that the two spins form an autonomous Hamiltonian system with two degrees of freedom [9].

Let us first investigate the classical system. The presence of the second quantum invariant $\hat{\sigma}_z$ in the zero-field xy interaction guarantees the integrability of the zero-field classical system. An initial search for a second classical invariant in the $\lambda\neq 0$ case suggests, however, that the two-spin system in a nonzero applied field is nonintegrable. To test this hypothesis, Poincaré surfaces of section (PSSs) were generated for increasing values of the magnetic field parameter $\lambda=0.02$, 0.2, 0.5, and 2.5 and are presented in Fig. 1. The four phase-space variables chosen are $(\theta_1,\theta_2,\phi_1,\phi_2)$ and the Poincaré surfaces of section were taken with $\theta_2=\pi/2$ and $\dot{\theta}_2>0$. The change of variables $\theta_i\rightarrow\pi-\theta_i$, $i=1,2$, leaves the energy and the sign of $\dot{\theta}_2$ unchanged, so this transformation is a symmetry of the Poincaré sections.

At $\lambda=0.02$, an elliptic fixed point appears at $(\phi_1\approx 2.4,\theta_1=\pi/2)$ and a hyperbolic fixed point appears at $(0.8,\pi/2)$. The unusual folded structure at the bottom and top of the PSS can be attributed to the singularity in the equation of motion for ϕ_1:

$$\dot{\phi}_1=\cot\theta_1\sin\theta_2\cos(\phi_1-\phi_2)-\lambda\cot\theta_1\sin\phi_1 \tag{3}$$

as $\theta_1\rightarrow 0$. At $\lambda=0.2$, there is a series of elliptic fixed points on the $\theta_1=\pi/2$ line and a prominent separatrix at the $(-0.8,\pi/2)$ fixed point. The circulation of trajectories around the empty elliptical regions near (0,0.2) and (0,2.9)

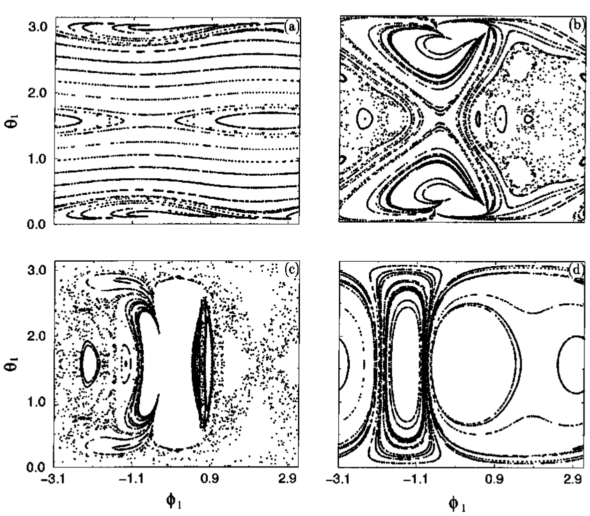

FIG. 1. Poincaré surfaces of sections for the xy model with increasing values of the magnetic field parameter λ. Each of the four sections is taken at energy $E=-0.1$ for a set of initial conditions spanning the $(\phi_1,\ \theta_1)$ plane: (a) $\lambda=0.02$, (b) $\lambda=0.2$, (c) $\lambda=0.5$, and (d) $\lambda=2.5$.

appears to be caused by the absence of solutions for the fourth phase-space variable ϕ_2 at the energy $E=-0.1$ in these regions. The scattering of points characteristic of chaos in Poincaré sections has clearly appeared in portions of this PSS.

With an increase of the field parameter to $\lambda=0.5$, the areas of chaos have become widespread, as has the region of points with no solution for the fourth phase-space variable. Finally, at $\lambda=2.5$, the section has returned to an almost integrable appearance, with continuous curves and no visible regions of scattered points. This may be understood as the parameter regime in which the magnetic field interaction begins to dominate the xy interaction, so that the PSS resembles the Poincaré section of the integrable model $H=S_{1x}+S_{2x}$. Overall, we observe the largest amount of classical chaos present for the parameter value $\lambda=0.5$ and we might expect that the signatures of quantum chaos would also be most apparent at this value.

Srivastava *et al.* [11], following a method due to Peres [10], established that an understanding of the changes in level spacing statistics characteristic of quantum chaos may be gained by creating a quantum web of simultaneous eigenstates of the Hamiltonian and a second commuting operator. In examining the quantum system, let us concentrate on the quantum chaotic effect of a disturbance in the structure of the quantum web of simultaneous eigenstates. In the case of the unperturbed system, the $\hat{\sigma}_z$ invariant defined above could be used as the second commuting operator, but in the model with nonzero applied field it is necessary to use Peres's idea of the time average of an operator that does not commute with the Hamiltonian as the second invariant in the system. Here the invariant $\sqrt{\hat{\sigma}_z^2}$ is selected, following the choice in the study of Srivastava *et al.* The level repulsion effects of quantum chaos take place between states of one given symmetry class of the geometrical symmetries of the system, which in this case consist of the parity operator $\hat{P}$ exchanging the two spins and the 180° spin-space rotation $\hat{C}_2^x(\pi)$. The quantum webs presented here therefore represent states from only a single symmetry class, that with eigenvalues $(1,-1)$ for the two geometrical symmetry operators.

The resulting quantum webs are pictured in Fig. 2. The spin magnitude $\sigma=20$ was chosen because it gave rise to the largest Hamiltonian matrix size that could be handled reasonably by the available computing power. The impression that smooth curves can be drawn through families of states in

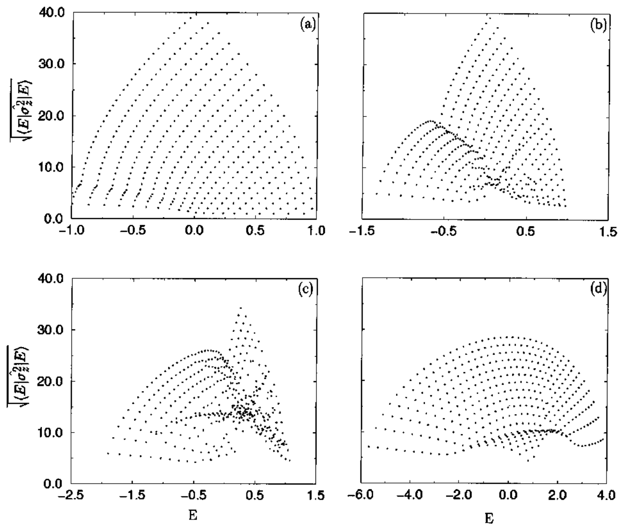

FIG. 2. Quantum webs of simultaneous eigenstates for the xy model with increasing values of the magnetic field parameter λ. The two interacting quantum spins have spin number $\sigma=20$: (a) $\lambda=0.02$, (b) $\lambda=0.2$, (c) $\lambda=0.5$, and (d) $\lambda=2.5$.

the quantum web is accurate and can be explained by the correspondence of classical tori and quantum eigenstates through semiclassical quantization [11]. We can clearly see that the amount of disruption of the regular web of states in the quantum web parallels closely the amount of chaos present in the PSS, increasing to a maximum at $\lambda=0.5$ before decreasing to a negligible amount at $\lambda=2.5$.

In conclusion, the Poincaré sections of two interacting classical spins in an applied magnetic field reflect the presence of chaos and the absence of a second classical invariant and the quantum webs of the corresponding quantum system show the signature of quantum chaos in the disruption of the regular pattern of the quantum web. This represents a clear effect of quantum chaos in a system related to large ferromagnetic lattices in applied magnetic fields, which are used in magnetic field sensors and hard-disk drives [12], and suggests that the effects of chaos in those larger systems may be most significant for intermediate applied fields.

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